A Plea for Adaptive Data Analysis:
Instantaneous Frequencies, Trends and degree of Non-stationary and Nonlinear

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Instantaneous Frequencies and Trends For Nonstationary Nonlinear Data

In search of frequency
I found the trend and other information,
e. g., quantification of nonlinearity

Prevailing Views on *Instantaneous Frequency*

The term, Instantaneous Frequency, should be banished forever from the dictionary of the communication engineer.

J. Shekel, 1953

The uncertainty principle makes the concept of an Instantaneous Frequency impossible.

K. Gröchennig, 2001

How to define frequency?

It seems to be trivial.

But frequency is an important parameter for us to understand many physical phenomena.

Definition of Frequency

Given the period of a wave as T ; the frequency is defined as

$$\omega = \frac{1}{T} .$$

Traditional Definition of Frequency

- **frequency = 1/period**.

- Definition too crude
- Only work for simple sinusoidal waves
- Does not apply to nonstationary processes
- Does not work for nonlinear processes
- Does not satisfy the need for wave equations

The Idea and the need of Instantaneous Frequency

According to the classic wave theory, the wave conservation law is based on a gradually changing ***φ(x,t)*** such that

$$\vec{k} = \nabla \varphi \ , \quad \omega = -\frac{\partial \varphi}{\partial t} \ ;$$

$$\Rightarrow \quad \frac{\partial \vec{k}}{\partial t} + \nabla \omega = 0 \ .$$

Therefore, both wave number and frequency must have instantaneous values and differentiable.

Instantaneous Frequency

$Velocity = \frac{distance}{time}$; *mean velocity*

$Newton \Rightarrow v = \frac{dx}{dt}$

$Frequency = \frac{1}{period}$; *mean frequency*

HHT defines the phase function $\Rightarrow \ \omega = \frac{d\theta}{dt}$

So that both v and ω can appear in differential equations.

Hilbert Transform : Definition

For any $x(t) \in L^p$,

$$y(t) = \frac{1}{\pi} \wp \int_{\tau} \frac{x(\tau)}{t-\tau} d\tau \ ,$$

then, $x(t)$ *and* $y(t)$ *form the analytic pairs:*

$$z(t) = x(t) + i\, y(t) = a(t)\, e^{i\theta(t)} \ ,$$

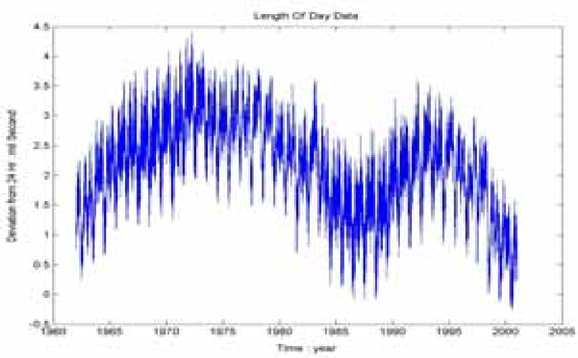
where

$$a(t) = \left(x^2 + y^2\right)^{1/2} \ \text{and} \ \theta(t) = \tan^{-1}\frac{y(t)}{x(t)} \ .$$

The Traditional View of the Hilbert Transform for Data Analysis

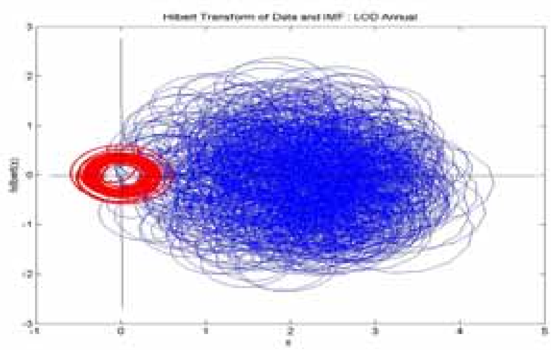
Traditional View

a la Hahn (1995) : Data LOD



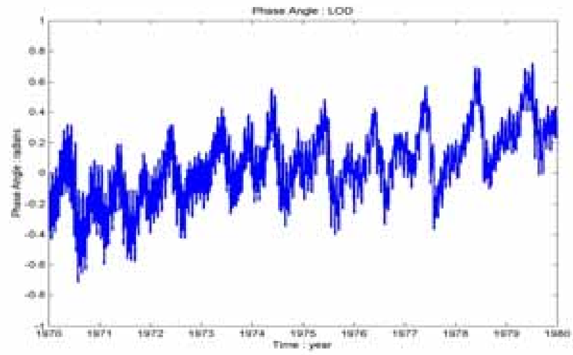
Traditional View

a la Hahn (1995) : Hilbert

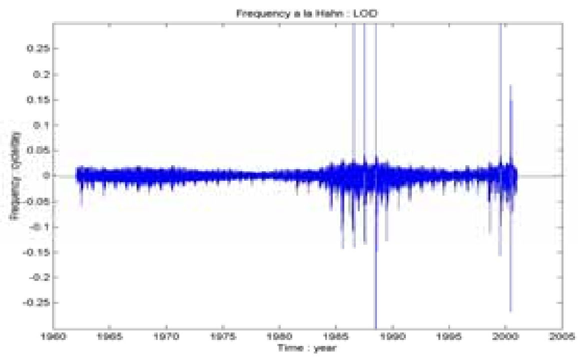


Traditional Approach
a la Hahn (1995) : Phase Angle

Traditional Approach
a la Hahn (1995) : Phase Angle Details



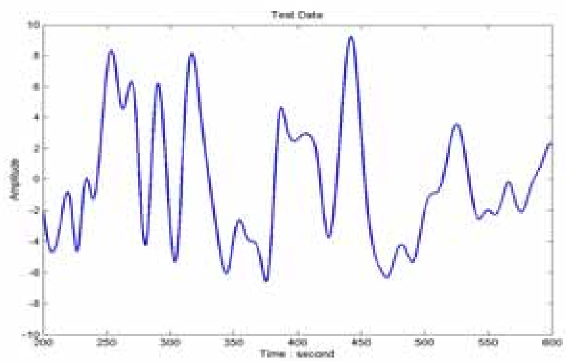
Traditional Approach
a la Hahn (1995) : Frequency



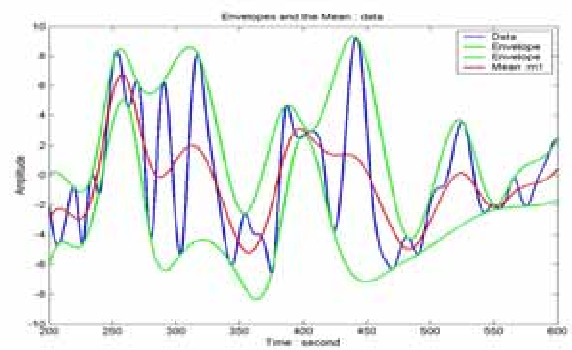
The Empirical Mode Decomposition
Method and Hilbert Spectral Analysis

Sifting

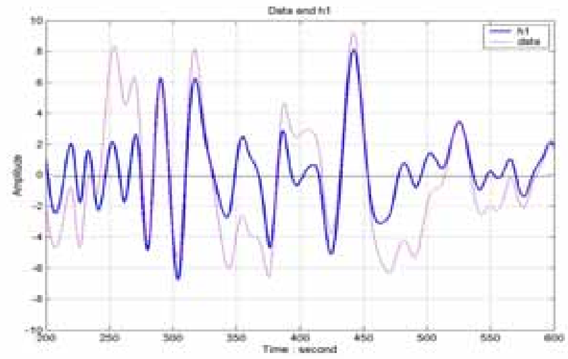
Empirical Mode Decomposition:
Methodology : Test Data



Empirical Mode Decomposition:
Methodology : data and m1



Empirical Mode Decomposition:
Methodology : data & h1



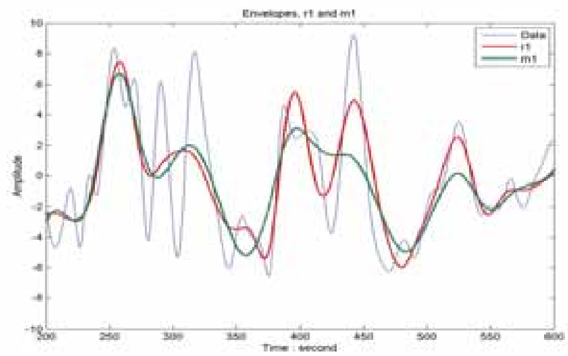
Definition of the Intrinsic Mode Function (IMF): a necessary condition only!

Any function having the same numbers of zero – cros sin gs and extrema, and also having symmetric envelopes defined by local max ima and min ima respectively is defined as an Intrinsic Mode Function (IMF).

All IMF enjoys good Hilbert Transform :

$$\Rightarrow\Rightarrow \quad c(t) = a(t)e^{i\theta(t)}$$

Empirical Mode Decomposition:
Methodology : data, r1 and m1



Empirical Mode Decomposition
Sifting : to get all the IMF components

$$x(t) - c_1 = r_1 \; ,$$
$$r_1 - c_2 = r_2 \; ,$$
$$\cdots$$
$$r_{n-1} - c_n = r_n \; .$$

$$\Rightarrow \; x(t) - \sum_{j=1}^{n} c_j = r_n \; .$$

Definition of Instantaneous Frequency

The Fourier Transform of the Instrinsic Mode Funnction, c(t), gives

$$W(\omega) = \int_t a(t)\, e^{i(\theta - \omega t)}\, dt$$

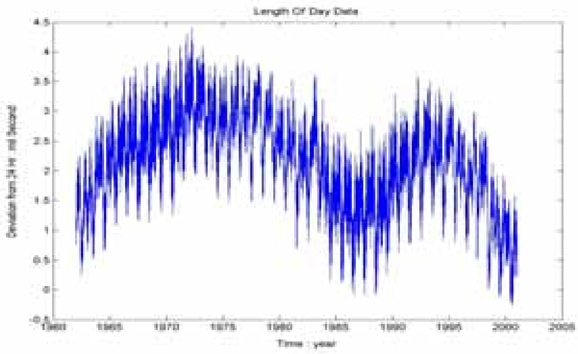
By Stationary phase approximation we have

$$\frac{d\theta(t)}{dt} = \omega \; ,$$

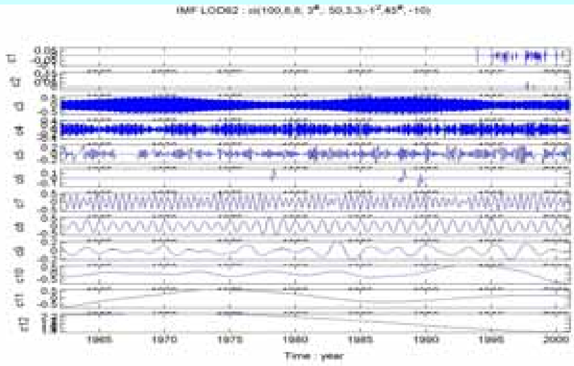
This is defined as the Ins tan tan eous Frequency.

An Example of Sifting & Time-Frequency Analysis

Length Of Day Data



LOD： IMF

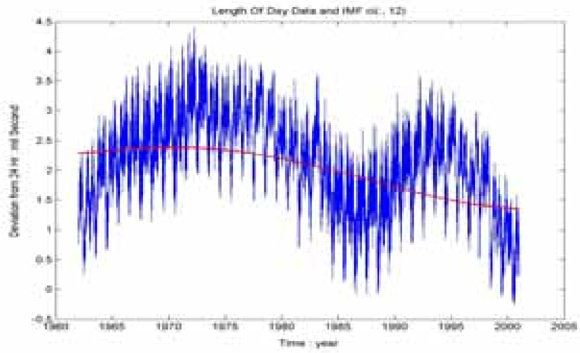


Orthogonality Check

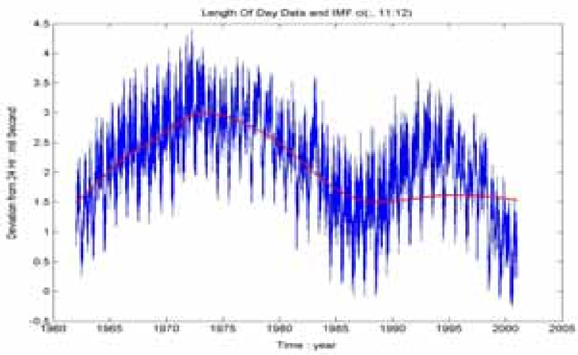
- Pair-wise %
- 0.0003
- 0.0001
- 0.0215
- 0.0117
- 0.0022
- 0.0031
- 0.0026
- 0.0083
- 0.0042
- 0.0369
- 0.0400

- Overall %
- 0.0452

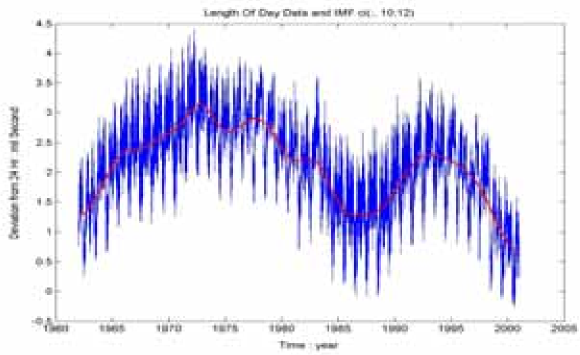
LOD：Data & c12



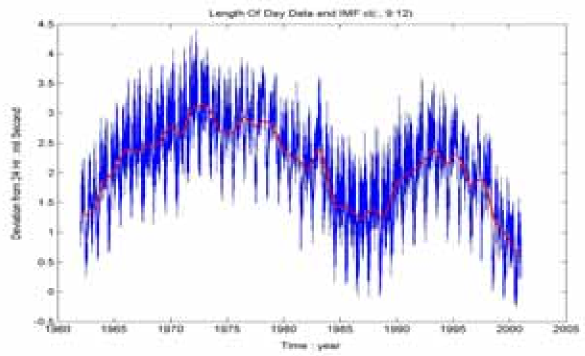
LOD：Data & Sum c11-12



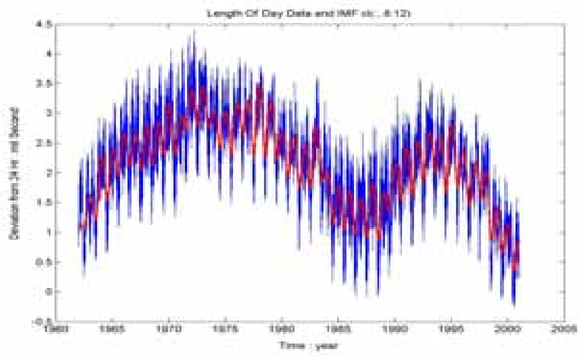
LOD：Data & sum c10-12



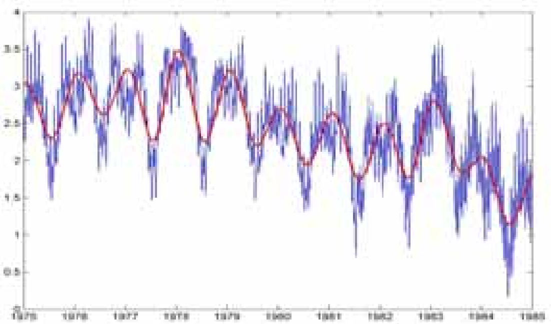
LOD : Data & c9 - 12



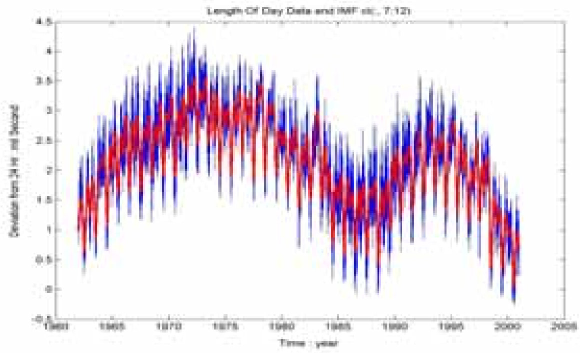
LOD : Data & c8 - 12



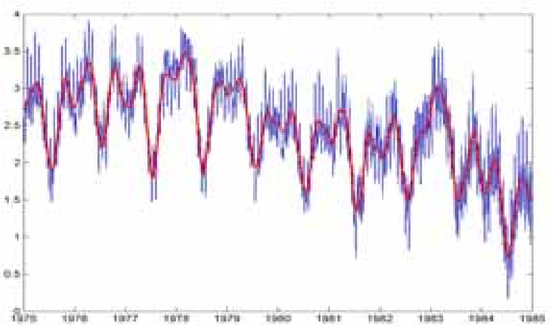
LOD : Detailed Data and Sum c8-c12



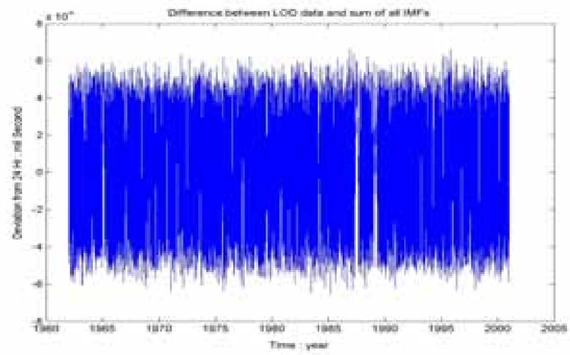
LOD : Data & c7 - 12



LOD : Detail Data and Sum IMF c7-c12



LOD : Difference Data – sum all IMFs



Properties of EMD Basis

The Adaptive Basis based on and derived from the data by the empirical method satisfy nearly all the traditional requirements for basis empirically and *a posteriori*:

- Complete
- Convergent
- Orthogonal
- Unique

The combination of Hilbert Spectral Analysis and Empirical Mode Decomposition has been designated by NASA as

HHT

(HHT vs. FFT)

Comparison between FFT and HHT

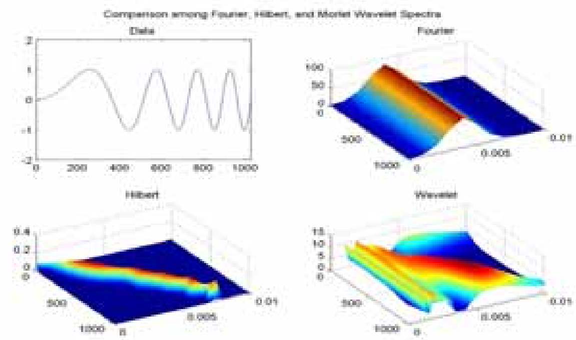
1. FFT:

$$x(t) = \Re \sum_j a_j e^{i \omega_j t} .$$

2. HHT:

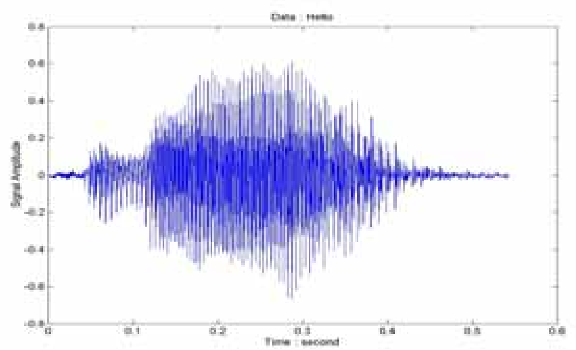
$$x(t) = \Re \sum_j a_j(t) e^{i \int_t \omega_j(\tau) d\tau} .$$

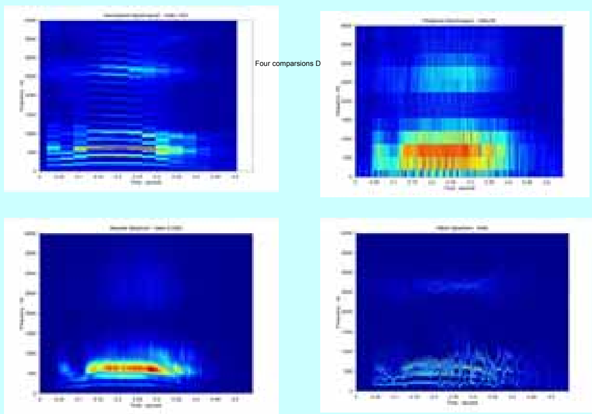
Comparisons: Fourier, Hilbert & Wavelet



Speech Analysis

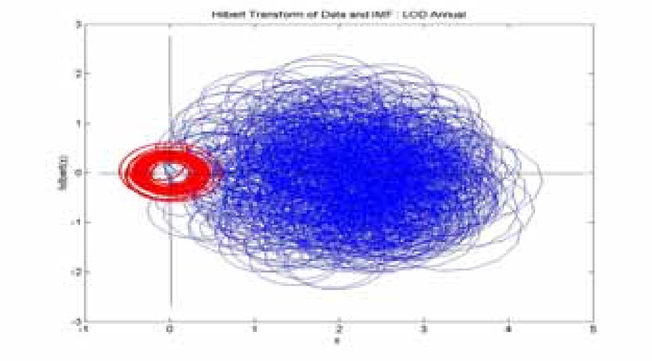
Hello : Data



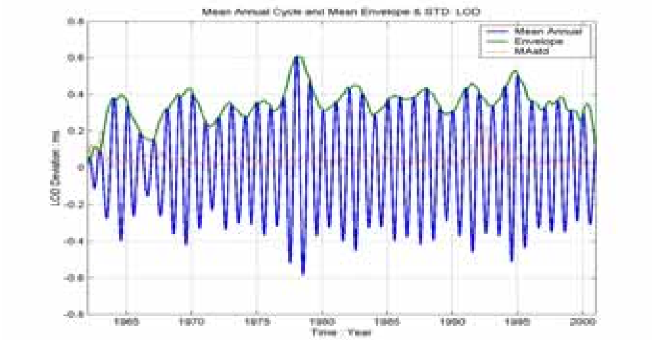


Traditional View

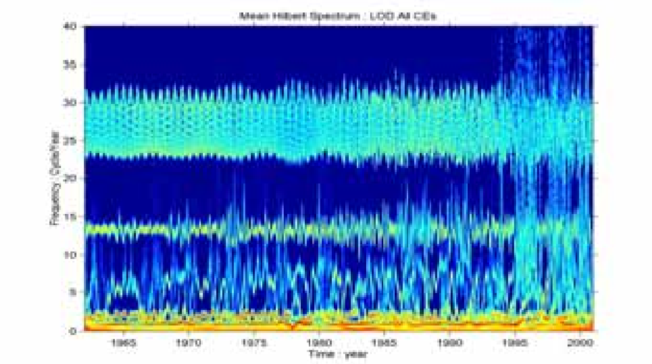
a la Hahn (1995) : Hilbert



Mean Annual Cycle & Envelope: 9 CEI Cases



Mean Hilbert Spectrum : All CEs



For

For quantifying nonlinearity we also need instantaneous frequency.

How to define Nonlinearity?

How to quantify it through data alone?

The term, ‘Nonlinearity,’ has been loosely used, most of the time, simply as a fig leaf to cover our ignorance.

Can we be more precise?

How is nonlinearity defined?

Based on Linear Algebra: nonlinearity is defined based on input vs. output.

But in reality, such an approach is not practical: natural system are not clearly defined; inputs and out puts are hard to ascertain and quantify. Furthermore, without the governing equations, the order of nonlinearity is not known.

In the autonomous systems the results could depend on initial conditions rather than the magnitude of the 'inputs.'

The small parameter criteria could be misleading: sometimes, the smaller the parameter, the more nonlinear.

Linear Systems

Linear systems satisfy the properties of superposition and scaling. Given two valid inputs to a system ***H***,

$$x_1(t) \text{ and } x_2(t)$$

as well as their respective outputs

$$y_1(t) = H\{x_1(t)\} \text{ and}$$
$$y_2(t) = H\{x_2(t)\}$$

then a linear system, ***H***, must satisfy

$$\alpha y_1(t) + \beta y_1(t) = H\{\alpha x_1(t) + \beta x_2(t)\}$$

for any scalar values **α** and **β**.

How is nonlinearity defined?

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The small parameter criteria could be misleading: sometimes, the smaller the parameter, the more nonlinear.

How should nonlinearity be defined?

The alternative is to define nonlinearity based on data characteristics: Intra-wave frequency modulation.

Intra-wave frequency modulation is known as the harmonic distortion of the wave forms. But it could be better measured through the deviation of the instantaneous frequency from the mean frequency (based on the zero crossing period).

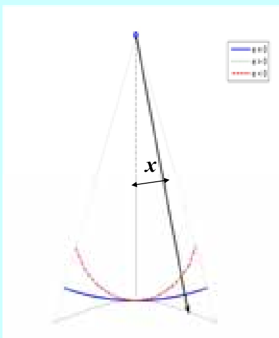
Characteristics of Data from Nonlinear Processes

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = \gamma \cos \omega t$$

$$\Rightarrow \frac{d^2x}{dt^2} + x\left(1 + \varepsilon x^2\right) = \gamma \cos \omega t$$

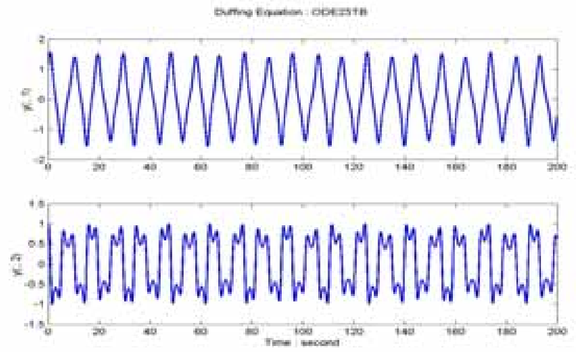
$\Rightarrow$ *Spring with position dependent constant, intra – wave frequency modulation; therefore, we need instantaneous frequency.*

Duffing Pendulum



$$\frac{d^2x}{dt^2} + x(1 + \varepsilon x^2) = \gamma \cos \omega t \ .$$

Duffing Equation : Data



Hilbert's View on Nonlinear Data

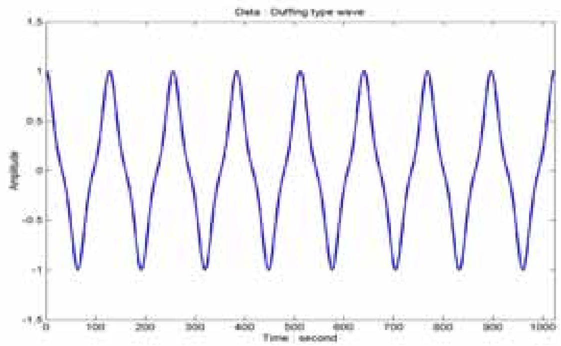
Intra-wave Frequency Modulation

A simple mathematical model

$$x(t) = \cos\left(\omega t + \delta \sin 2\omega t\right)$$

Duffing Type Wave

Data: $x = \cos(wt+0.3 \sin 2wt)$



Duffing Type Wave

Perturbation Expansion

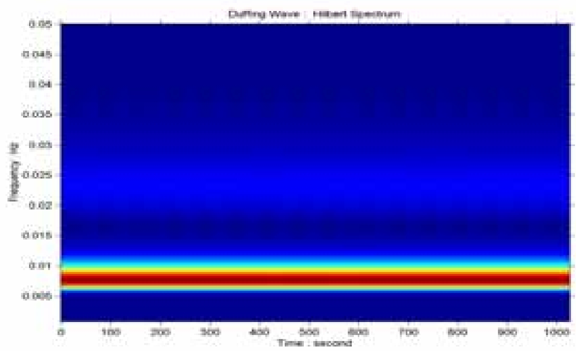
For $\delta \ll 1$, *we can have*

$$\begin{aligned} x(t) &= \cos\left(\omega t + \delta \sin 2\omega t\right) \\ &= \cos \omega t \cos\left(\delta \sin 2\omega t\right) - \sin \omega t \sin\left(\delta \sin 2\omega t\right) \\ &= \cos \omega t - \delta \sin \omega t \sin 2\omega t + \ldots. \\ &= \left(1 - \frac{\delta}{2}\right)\cos \omega t + \frac{\delta}{2}\cos 3\omega t + \ldots. \end{aligned}$$

This is very similar to the solution of Duffing equation .

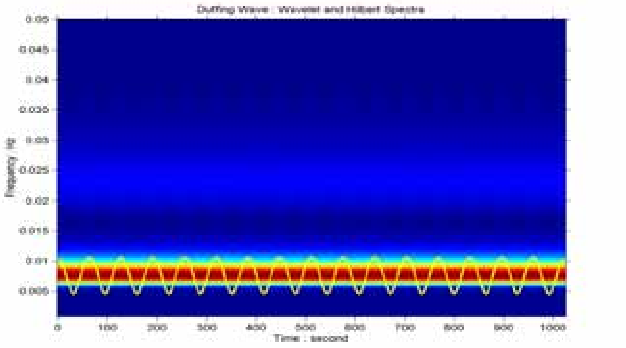
Duffing Type Wave

Wavelet Spectrum



Duffing Type Wave

Hilbert Spectrum



Degree of nonlinearity

Let us consider a generalized intra-wave frequency modulation model as:

$$x(t)=\cos(\omega t+\delta\sin\eta\omega t)\ \Rightarrow\ IF=\frac{d\theta}{dt}=\omega(1+\eta\delta\cos\eta\omega t)$$

DN (Degree of Nolinearity) should be $\propto\left\langle\left(\frac{IF-IF_z}{IF_z}\right)^2\right\rangle^{1/2}=\frac{\eta\delta}{\sqrt{2}}.$

Depending on the value of η, we can have either a up-down symmetric or a asymmetric wave form.

Degree of Nonlinearity

- DN is determined by the combination of $\delta\eta$ precisely with Hilbert Spectral Analysis. Either of them equals zero means linearity.
- We can determine δ *and* η separately:
  - η can be determined from the instantaneous frequency modulations relative to the mean frequency.
  - δ can be determined from DN with known η.

  NB: from any IMF, the value of $\delta\eta$ cannot be greater than 1.
- ***The combination of δ and η gives us not only the Degree of Nonlinearity, but also some indications of the basic properties of the controlling Differential Equation, the Order of Nonlinearity.***

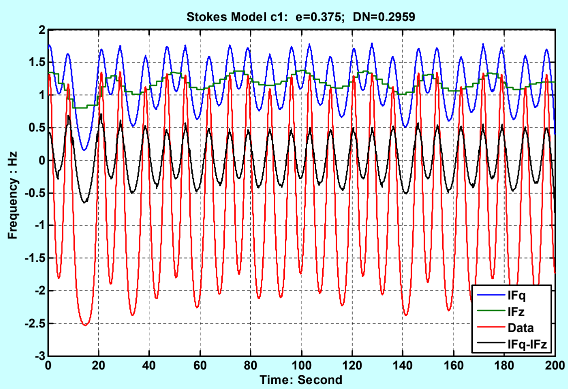
Stokes Models

$$\frac{d^2x}{dt^2}+x+\varepsilon x^2=\gamma\cos\omega t\quad \text{with}\ \omega=\frac{2\pi}{25};\ \gamma=0.1.$$

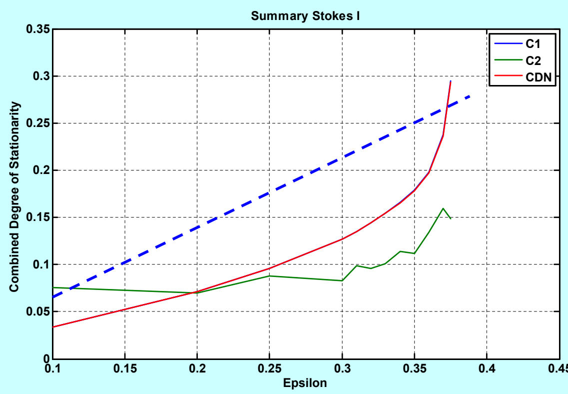
Stokes I: ***ε is positive ranging from 0.1 to 0.375; beyond 0.375, there is no solution.***

Stokes II: ***ε is negative ranging from 0.1 to 0.391; beyond 0.391, there is no solution.***

Data and IFs : C1



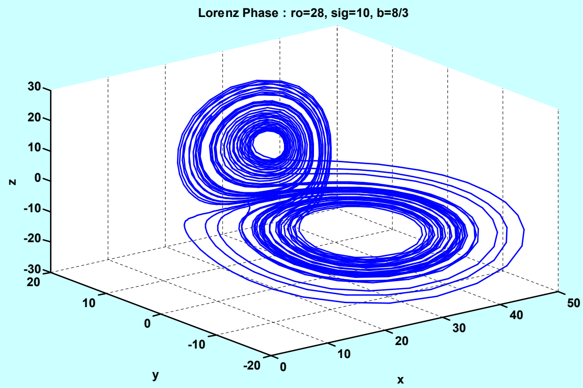
Summary Stokes I



Lorenz Model

- Lorenz is highly nonlinear; it is the model equation that initiated chaotic studies.
- Again it has three parameters. We decided to fix two and varying only one.
- There is no small perturbation parameter.
- We will present the results for ρ=28, the classic chaotic case.

Phase Diagram for ro=28

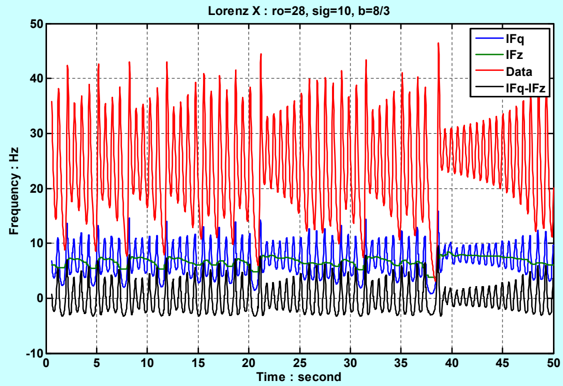


X-Component

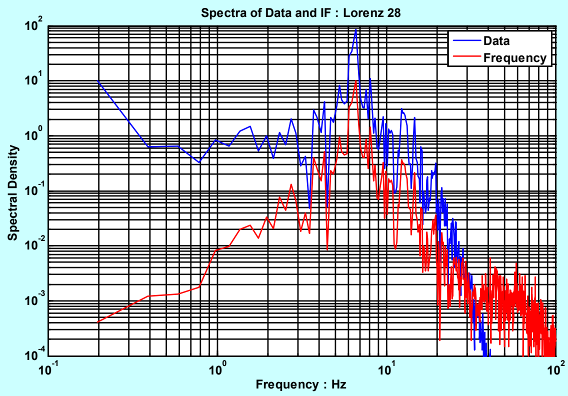
DN1=0.5147

CDN=0.5027

Data and IF



Spectra data and IF



Comparisons

	Fourier	Wavelet	Hilbert
Basis	a priori	a priori	Adaptive
Frequency	Integral transform: Global	Integral transform: Regional	Differentiation: Local
Presentation	Energy-frequency	Energy-time-frequency	Energy-time-frequency
Nonlinear	no	no	yes, quantifying
Non-stationary	no	yes	Yes, quantifying
Uncertainty	yes	yes	no
Harmonics	yes	yes	no

How to define Trend ?

Parametric or Non-parametric?
Intrinsic vs. extrinsic approach?

The State-of-the arts: Trend

"One economist's trend is another economist's cycle"

Watson : Engle, R. F. and Granger, C. W. J. 1991 *Long-run Economic Relationships*. Cambridge University Press.

Philosophical Problem Anticipated

名不正則言不順

言不順則事不成

——孔夫子

On Definition

Without a proper definition,
logic discourse would be impossible.
Without logic discourse,
nothing can be accomplished.

Confucius

Definition of the Trend

Within the given data span, the trend is an intrinsically fitted monotonic function, or a function in which there can be at most one extremum.

The trend should be an intrinsic and local property of the data; it is determined by the same mechanisms that generate the data.

Being local, it has to associate with a local length scale, and be valid only within that length span, and be part of a full wave length.

The method determining the trend should be intrinsic. Being intrinsic, the method for defining the trend has to be adaptive.

All traditional trend determination methods are extrinsic.

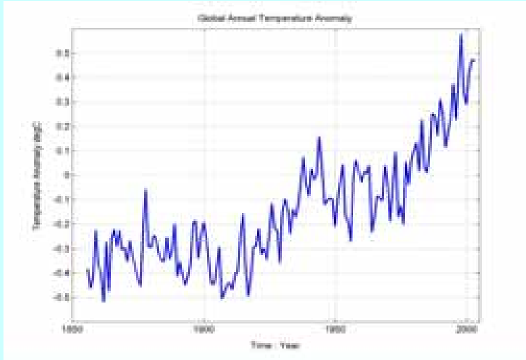
Algorithm for Trend

- Trend should be defined neither parametrically nor non-parametrically.
- It should be the residual obtained by removing cycles of all time scales from the data intrinsically.
- Through EMD.

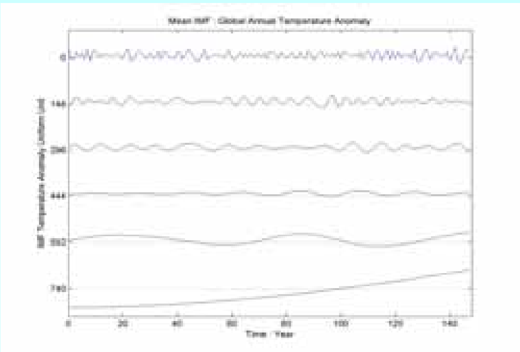
Global Temperature Anomaly

Annual Data from 1856 to 2003

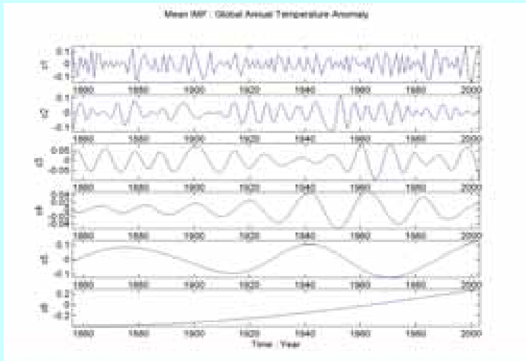
Global Temperature Anomaly 1856 to 2003



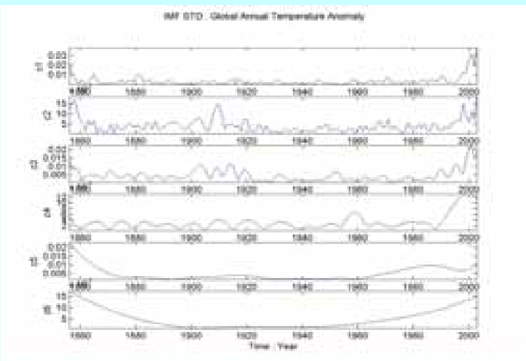
IMF Mean of 10 Sifts : CC(1000, I)



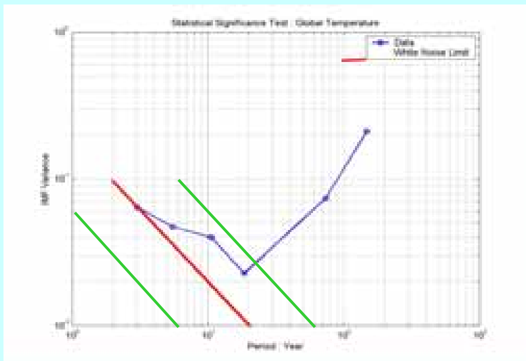
Mean IMF



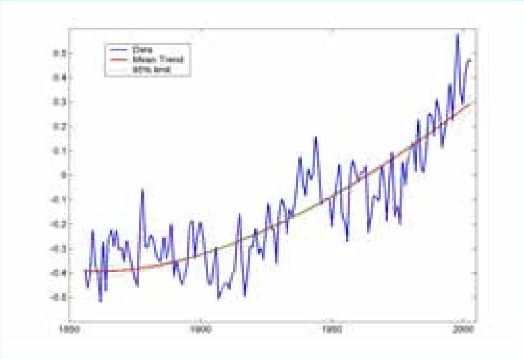
STD IMF



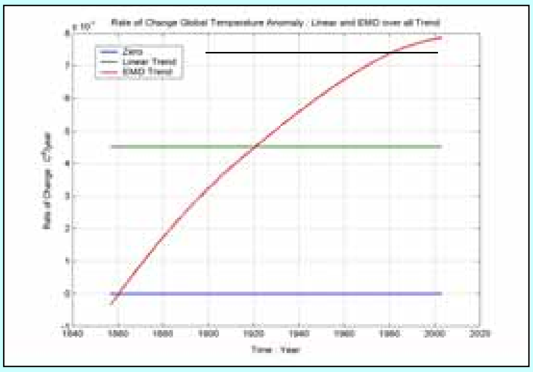
Statistical Significance Test



Data and Trend C6



Rate of Change Overall Trends : EMD and Linear



Conclusion

- With EMD, we can define the true instantaneous frequency and extract trend from any data.
- We can also talk about nonlinearity quantitatively.
- Among other applications, the degree of nonlinearity could be used to set an objective criterion in structural health monitoring and to quantify the degree of nonlinearity in natural phenomena; the trend could be used in financial as well as natural sciences.
- These are all possible because of adaptive data analysis method.

The Job of a Scientist

The job of a scientist is to listen carefully to nature, not to tell nature how to behave.

Richard Feynman

To listen is to use adaptive method and let the data sing, and not to force the data to fit preconceived modes.

All these results depends on adaptive approach.

Thanks